

Noname manuscript No.
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Spin-structures of N-boson systems with nonzero spins

- an analytically solvable model with pairing force

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Received: date / Accepted: date

Abstract A model is proposed to study the possible pairing structures of N-boson systems with nonzero spin. Analytical solutions have been obtained. The emphasis is placed on the spin-structures of ground states with attractive or repulsive pairing force, and with or without the action of a magnetic field. A quantity (an analogue of the two-body density function) is defined to study the spin-correlation between two bosons in N-body systems. The excitation of the system has also been studied.

1 Introduction

Historically, the pairing structures played a very important role in fermion systems. The most famous example is the Cooper-pair in superconductivity. [1] One more example is the pairing of the valence nucleons in nuclei as described by the "Interacting Boson Model" [2]. However, there is no obvious evidence to support the existence of paring structures in boson systems. Nonetheless, we can not rule out the possibility. For an example, each ^{52}Cr atom has a spin $J = 3$. The S-wave scattering lengths a_s of these atoms depend on the spin-channels s , namely, the total spin of the two interacting atoms. There are four channels, $s = 0, 2, 4$, and 6. Accordingly, for low energy scattering, the interaction between the i and j atoms can be written as

$$v_{ij} = \delta(\mathbf{r}_i - \mathbf{r}_j) \sum_s g_s \mathfrak{P}_{ij}^s \quad (1)$$

where the strength $g_s = 4\pi\hbar^2 a_s/M$, M is the mass of atom. \mathfrak{P}_{ij}^s is the projection operator of the s -channel. Let $\chi(i)$ be the spin-state of i , $(\chi(i)\chi(j))_{sm} \equiv (ij)_{sm}$ be the normalized total spin-state of i and j , where the spins are coupled to s and m (the Z-component of s) via the Clebsch-Gordan coefficients. Then, $\mathfrak{P}_{ij}^s \equiv \sum_m |(ij)_{sm}\rangle\langle(ij)_{sm}|$. For ^{52}Cr , it has been determined experimentally that $a_2 =$

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$-7a_B$, $a_4 = 58a_B$, and $a_6 = 112a_B$, while a_0 is undetermined.[3] Since g_4 and g_6 are strongly repulsive, the atoms in low-lying states would do their best to avoid forming $s = 4$ and 6 pairs. Although g_2 is attractive, it is weak. Therefore, if g_0 is also attractive and strong, the low-lying states might be dominated by the force of the $s = 0$ channel (the pairing force) to form $s = 0$ pairs. Therefore, in view of having so many different kinds of boson systems (the constituent may be an atom, a pair of fermions, an exciton, etc.), the assumption that some boson-systems in some cases might be dominated by pairing force is assumable.

In what follows, we shall propose a model to study the effect of the pairing force on boson systems. In this model, each particle has a nonzero integral spin J , and they are interacting via pairing interaction. In general, if one can obtain analytical solutions of a Hamiltonian, then these solutions are valuable because the underlying physics can be thereby better understood. Besides, as an exact solution, it can be used to check the validity of various approximate methods. It turns out that the above model can be solved analytically as shown below.

Since the experimental realization of the condensation of atoms with nonzero spins,[4] the study of the spinor condensates has become a hot topic due to academic interest and their potential in application. Traditionally, the mean-field theory is used to explain related phenomena. Up to now this theory is successful. Nonetheless, if the details of spin-spin correlation are taken into account, this theory is not sufficient due to the basic approximation inherent in the theory. On the other hand, the theoretical method used in this paper is beyond the mean-field theory. In what follows, approximation is made only in the spatial space, while the spin-degrees of freedom of the N -body system have been exactly treated. Thus, the following approach would be a useful complement to the mean-field theory. The spin-spin correlation might be thereby better understood.

2 Hamiltonian, spectrum, and the eigenstates

We first consider a N -boson system, each boson has spin $J = 3$. The cases with $J \neq 3$ will be discussed later. The bosons are confined by a potential. The interaction acting among the bosons contains a spin-independent term and a spin-dependent term. The latter is simply the pairing force, namely,

$$v_{ij} = \delta(\mathbf{r}_i - \mathbf{r}_j) g_0 \mathfrak{P}_{ij}^0 . \quad (2)$$

We are only interested in the case of the low-temperature limit. At this limit all particles are condensed into the same spatial state $f(\mathbf{r})$.[5] In other words, the spatial state is fixed. Therefore, the spatial degrees of freedom can be integrated. After the integration, the Hamiltonian of the model reads

$$H_{mod} = \sum_{i < j} G_0 \mathfrak{P}_{ij}^0 \quad (3)$$

where $G_0 \equiv g_0 \int d\mathbf{r} |f(\mathbf{r})|^4$. For this Hamiltonian, only the spin-degrees of freedom are considered. Obviously, $f(\mathbf{r})$ depends on the potential of confinement and on the interaction. However, since G_0 will be considered as a parameter, the details of the potential and the spin-independent interaction is irrelevant. In what follows, g_0 (and therefore G_0) is first considered to be negative.

For the diagonalization of H_{mod} , we introduce the set of symmetrized and normalized Fock-states $|k\rangle \equiv |N_3^k, N_2^k, N_1^k, N_0^k, N_{-1}^k, N_{-2}^k, N_{-3}^k\rangle$ as basis functions, where N_μ^k is the number of atoms with spin component μ , and k denotes the set $\{N_\mu^k\}$ which together determine the basis function. Obviously, $\sum_\mu N_\mu^k = N$, the total number of particles. Since only the spin-degrees of freedom are involved in H_{mod} , the set of Fock-states is complete. The related matrix elements read

$$\begin{aligned} \langle k'|H_{mod}|k\rangle &= \frac{G_0}{14} \sum_{\mu,\nu} (-1)^{\mu+\nu} \sqrt{N_\mu^k N_\nu^{k'}} [\delta_{\nu,0}\delta_{\mu,0}\delta_{k'k}(N_0^k - 1) \\ &\quad + \bar{\delta}_{\nu,0}\delta_{\mu,0}\sqrt{N_{-\nu}^{k'}(N_0^k - 1)}\delta_{N_0^k-2}^{N_\nu^{k'}-1, N_{-\nu}^{k'}-1} \\ &\quad + \delta_{\nu,0}\bar{\delta}_{\mu,0}\sqrt{(N_0^{k'} - 1)N_{-\mu}^k}\delta_{N_\mu^k-1, N_{-\mu}^k-1}^{N_\nu^{k'}-2} \\ &\quad + \bar{\delta}_{\nu,0}\bar{\delta}_{\mu,0}\sqrt{N_{-\nu}^{k'}N_{-\mu}^k}\delta_{N_\mu^k-1, N_{-\mu}^k-1}^{N_\nu^{k'}-1, N_{-\nu}^{k'}-1}] \end{aligned} \quad (4)$$

where $\bar{\delta}_{\mu,\nu} \equiv (1 - \delta_{\mu,\nu})$. For the label $\delta_{N_0^k-2}^{N_\nu^{k'}-1, N_{-\nu}^{k'}-1}$, the superscript implies a revised set of the set $\{N_3^k \dots N_{-3}^k\}$ by reducing both $N_\nu^{k'}$ and $N_{-\nu}^{k'}$ by 1, the subscript implies a revised set of $\{N_3^k \dots N_{-3}^k\}$ by reducing N_0^k by 2. When the two revised sets are one-to-one identical, the label is 1, otherwise it is zero. And so on. When N is given, the dimension of the matrix is finite. Due to the special structure of the matrix, it can be analytically diagonalized. The results are the follows.

For convenience, N is assumed to be even unless specified. The ground state Ψ_g has an analytical form as

$$\Psi_g = \sum_k C_k^g |k\rangle, \quad (\text{not yet normalized}) \quad (5)$$

$$C_k^g = \delta_{N_3^k, N_{-3}^k} \delta_{N_2^k, N_{-2}^k} \delta_{N_1^k, N_{-1}^k} (-1)^{N_3^k + N_1^k} \sqrt{N_0^k! / ((N_0^k/2)! 2^{N_0^k/2})} \quad (6)$$

where N_0^k should be even. On the other hand, we define

$$\Psi_0^{pair} \equiv \mathbf{P}\{(12)_0(34)_0 \dots (N-1, N)_0\} \quad (7)$$

where \mathbf{P} implies a summation over all the $N!$ permutations of particles, and the implication of $(ij)_0$ has been given in the previous section. This state can be expanded in terms of the Fock-states as

$$\Psi_0^{pair} \equiv \frac{(-1)^{N/2}}{7^{N/4}} (N/2)! 2^{N/2} \sqrt{N!} \sum_k C_k^g |k\rangle. \quad (8)$$

Hence, Ψ_0^{pair} is just the ground state that we have obtained from the diagonalization. Thus the structure of the ground state is clear, it is just a product of the $s = 0$ pairs, i.e., a pairing structure.

We use $|G_0|$ as the unit of energy in the follows, then the ground state energy turns out to be $E_g = -\frac{N(N+5)}{14} \equiv \varepsilon_N$.

If N is odd, the ground state would be just the above pairing structure together with an additional particle, and $E_g = \varepsilon_N + 3/7$.

It is reminded that the interaction keeps the total spin S and its Z-component M to be conserved. Obviously, due to the pairing structure, the ground state has $S = 0$. The first excited level is found to contain the eigen-states having $S = 2, 4$, and 6 . They are exactly degenerate. Their analytical forms read

$$\Psi_{y,SM} = \sum_k C_k^{y,SM} |k\rangle, \quad (\text{not yet normalized}) \quad (9)$$

$$C_k^{y,SM} = \sum_\nu C_{3,\nu;3,M-\nu}^{SM} F_{k\nu} \quad (10)$$

where the script y implies that the state is a yrast state, and the Clebsch-Gordan coefficients have been introduced, and

$$F_{k\nu} = \delta_{L_3,L_{-3}} \delta_{L_2,L_{-2}} \delta_{L_1,L_{-1}} (-1)^{L_3+L_1} \sqrt{L_0!} / ((L_0/2)! 2^{L_0/2}) \\ \times [\delta_{\nu,M-\nu} \sqrt{(L_\nu+2)(L_\nu+1)} + \bar{\delta}_{\nu,M-\nu} \sqrt{(L_\nu+1)(L_{M-\nu}+1)}] \quad (11)$$

where L_μ and N_μ^k are related as $L_\mu = N_\mu^k - \delta_{\mu,\nu} - \delta_{\mu,M-\nu}$ (μ is from -3 to 3) and L_0 should be even.

Similarly, we define an excited pairing structure by changing a $s = 0$ pair to a $s \neq 0$ pair. We found

$$P\{(12)_{SM}(34)_0 \cdots (N-1,N)_0\} \equiv \text{cons} \times \sum_k C_k^{y,SM} |k\rangle. \quad (12)$$

Thus, it is clear that the $\Psi_{y,SM}$ states have just the excited pairing structures. They all have the same eigenenergy $E_g + 1$ disregarding S and N , and the gap is just 1 (in $|G_0|$, [6]).

In general, the spectrum of this N -body system contains a finite number of levels, each is highly degenerate (except the lowest one). We introduce a quantum number I to denote the levels, therefore I describes the degree of excitation. Each eigenenergy is a sum of two terms, it reads

$$E_I = \varepsilon_N + \varepsilon_I. \quad (13)$$

Where $\varepsilon_N = -\frac{N(N+5)}{14}$ depends roughly on the number of pairs ($\sim N^2$) and is the same for all the eigen-states, while $\varepsilon_I = \frac{I(I+5)}{14}$ depends on I , $I = 0, 2, 4, \dots, N$ (if N is even, in this case the first excited level has $I = 2$ and therefore $\varepsilon_I = 1$), or $I = 1, 3, \dots, N$ (if N is odd). So, when N is even (odd), totally there are $N/2 + 1$ ($(N+1)/2$) energy levels. When I is larger, the level will be not only higher, but also have a larger degeneracy as shown later. When we compare the spectra of two systems distinct in N , their ε_I are one-to-one identical (except the range of I). Thus, although the spectrum of the one with more particles would contain more levels, the lower parts of their spectra are exactly the same except the shift caused by ε_N . This is a noticeable point. The spacing of adjacent levels is just $E_{I+2} - E_I = (2I+7)/7$. Thus, when I goes up, the spacing becomes larger and larger. Note that, since the set of Fock-states is complete, the above spectrum is exact for the model Hamiltonian.

For all the states of the I-level, I atoms would be excited from the $s = 0$ pairs. Therefore, I has a similar implication as the seniority which has been introduced for describing the excitation of pairing structures of fermion systems long ago.[7,8] Furthermore, the maximum S of the states belonging to an I-level is obvious $3I$ contributed by the I excited atoms.

In general, the eigen-states can be classified according to S , M , and I . It is possible that there are more than one states having the same set (ISM) (e.g., for the $I = 4$ level, there are two independent $S = 4$ states). In this case an extra quantum number is necessary. Therefore, an eigenstate can be labeled as $\Psi_{I,S,M,i}$, where i denotes other good quantum numbers, if necessary. For examples, the above three yrast states $\Psi_{y,SM}$ have $I = 2$, therefore they can also be denoted as $\Psi_{2,SM}$.

The degeneracy of an I -level is stated as follows. Let $\{j_\mu\}$ be a set of seven non-negative integers (μ is from -3 to 3). There would be a finite number of sets satisfy both $\sum_\mu j_\mu = I$ and $\sum_\mu \mu j_\mu = 0$. The number of these sets is denoted by Ω_I . Then, one can prove that the degeneracy of the I -th level $d_I = \Omega_I - \Omega_{I-2}$ ($d_0 = d_1 = 1$). This number increases very fast with I , e.g., $d_0 = 1$, $d_2 = 3$, and $d_4 = 14$. Note that the degeneracy caused by M has not yet been taken into account.

For higher excitation, a state from the $I = 4$ level is given as an example as

$$\Psi_{400} = \text{cons} \times (3P\{[(12)_2(34)_2]_0(56)_0 \cdots\} - \sqrt{5}P\{[(12)_4(34)_4]_0(56)_0 \cdots\}) \quad (14)$$

where four particles are excited from the $s = 0$ pairs. The structures of higher states will become more and more complicated because more particles are excited. We are not going to the details of them.

When G_0 is positive, the above analytical solutions hold exactly. However, the spectrum would be reversed. The ground states would be highly degenerate, all of them belong to the $I = N$ level, wherein all $s = 0$ pairs are rejected. In particular, there is a great energy gap $E_{gap} = E_{N-2} - E_N = (2N + 3)/7$. A larger N leads to a larger gap. Thus, when N is large, the spin-degrees of freedom of the group of ground states are difficult to be excited. This is also a noticeable point.

When $J \neq 3$, one can also use the Fock-states as basis functions. After the diagonalization, the eigenenergies are also a sum of two terms, namely, $E_I = \varepsilon_N + \varepsilon_I$. However, $\varepsilon_N = -\frac{N(N+2J-1)}{2(2J+1)}$, while $\varepsilon_I = \frac{I(I+2J-1)}{2(2J+1)}$. The eigenstates have the same structures as before, and therefore can also be classified as Ψ_{ISMi} . In particular, the $I = 0$ level has all particles in $s = 0$ pairs, while the $I = 2$ level has two particles excited from the $s = 0$ pairs, and so on.

3 Ground states under a magnetic field

We go back again to the cases with $J = 3$ and an even N and a negative G_0 . When a magnetic field B lying along the Z-axis is applied, the model Hamiltonian reads (in $|G_0|$)

$$H'_{mod} = - \sum_{i < j} \mathfrak{P}_{ij}^0 - \gamma \hat{S}_Z \quad (15)$$

where \hat{S}_Z is the operator of the total spin along the Z-axis, $\gamma = g\mu_B B/|G_0|$, μ_B is the Bohr magneton (incidentally, if the atom is ^{52}Cr with $J = 3$, $g = 2$). The eigen-states of H'_{mod} are exactly the same as H_{mod} , however the eigen-energies would contain an additional term due to the Stern-Gerlach splitting as

$$E'_{ISM} = \varepsilon_N + \varepsilon_I - \gamma M . \quad (16)$$

We are interested in how the structure of the ground state would vary with B . For the group of states belonging to the I -level, the largest M possessed by these states

is $3I$. Thus the state with $S = M = 3I$ of each I -level is a candidate of the ground state with the energy $E'_{I,3I,3I}$. Among the candidates, when B (or γ) is fixed, if $I = I_o$ leads to the minimum of the group $E'_{I,3I,3I}$, then the ground state would have $I = I_o$. Obviously, I_o depends on B . It was found that, when B is small so that $\gamma < 7/42$, the ground state has $I_o = 0$. Starting from $7/42$, let the range of γ be divided into segments, each has a length $2/21$. Then, when γ increases, I_o will increase step by step from a segment to the next segment. Each step I_o will jump by 2. The general relation of I_o and γ is:

$$(2I_o + 3)/42 \leq \gamma < (2I_o + 7)/42. \quad (17)$$

Since I_o can not exceed N , the last segment is $(2N + 3)/42 \leq \gamma < \infty$. In this extensive segment, $I_o = N$, namely, the increase of the magnetic field will eventually excite all particles from the $s = 0$ pairs.

In each segment the associated ground state energy $E_{gr}(\gamma) = E'_{I_o,3I_o,3I_o}$, and $\frac{d}{d\gamma}E_{gr}(\gamma) = -3I_o$ within the segment. When γ increases, since I_o will increase step by step, the slope becomes more and more negative.

When $\gamma < (2N + 3)/42$, we have approximately $\gamma \approx (2I_o + 5)/42$. Thus we have $E_{gr}(\gamma) \approx -\varepsilon_N - (42\gamma - 5)^2/56$. It implies that, when γ increases, the decline of $E_{gr}(\gamma)$ against γ is nearly parabolic. Of course, when $\gamma > (2N + 3)/42$, the decline is only contributed by the last term of eq.(16), i.e., $-3\gamma N$, and therefore is linear.

The analytical form of the wave function of the ground state can also be obtained. It reads in terms of the Fock-states as (not yet normalized)

$$\begin{aligned} \Psi_{gr}(\gamma) \equiv \Psi_{I_o,3I_o,3I_o} &= \sum_k \delta_{N_3^k - I_o, N_{-3}^k} \delta_{N_2^k, N_{-2}^k} \delta_{N_1^k, N_{-1}^k} (-1)^{N_3^k + N_1^k} \sqrt{N_0^k! N_3^k! / N_{-3}^k!} \\ &\quad / ((N_0^k/2)! 2^{N_0^k/2}) |k\rangle \\ &= cons \times P\{[\prod_{i=1}^{I_o} \chi_3(i)](I_o + 1, I_o + 2)_0 \cdots (N - 1, N)_0\}. \end{aligned} \quad (18)$$

Thus, when a magnetic field is applied, the ground state would have a portion of particles excited and become fully polarized, while the others remain in the $s = 0$ pairs. Thus the implication of I_o is clear, it is just the number of polarized particles.

From $\Psi_{gr}(\gamma)$ it is straight forward to calculate the probability of a particle in μ -component $P_\mu \equiv \langle \Psi | \hat{N}_\mu | \Psi \rangle / N$, where \hat{N}_μ is the operator of the number of particles in μ , and $\sum_\mu P_\mu = 1$. Obviously, if $P_3 = 1$ (or $P_{-3} = 1$) while the other $P_\mu = 0$, the system is fully polarized. Therefore, from P_μ one can understand the situation of magnetization of a state. For $\Psi_{gr}(\gamma)$, from (18) we have

$$\begin{aligned} P_3 &= \frac{I_o(I_o + N + 6) + N}{N(7 + 2I_o)} \\ P_{-3} &= \frac{(I_o + 1)(N - I_o)}{N(7 + 2I_o)} \\ P_\mu &= \frac{N - I_o}{N(7 + 2I_o)}. \quad (\text{if } \mu \neq \pm 3) \end{aligned} \quad (19)$$

Examples of P_μ against γ are given in Fig.1. When $\gamma = 0$, all $P_\mu = 1/7$. In this case all particles are in $s = 0$ pairs. Accordingly, the total spin-state has $S = 0$ and therefore is isotropic. When γ increases, the variation of P_μ can be roughly divided into three stages. In the first stage both P_3 and P_{-3} increases very fast while the other P_μ decreases very fast as shown in the figure. When $\gamma = 93N/(21(2N + 100)) \equiv \gamma_1$,

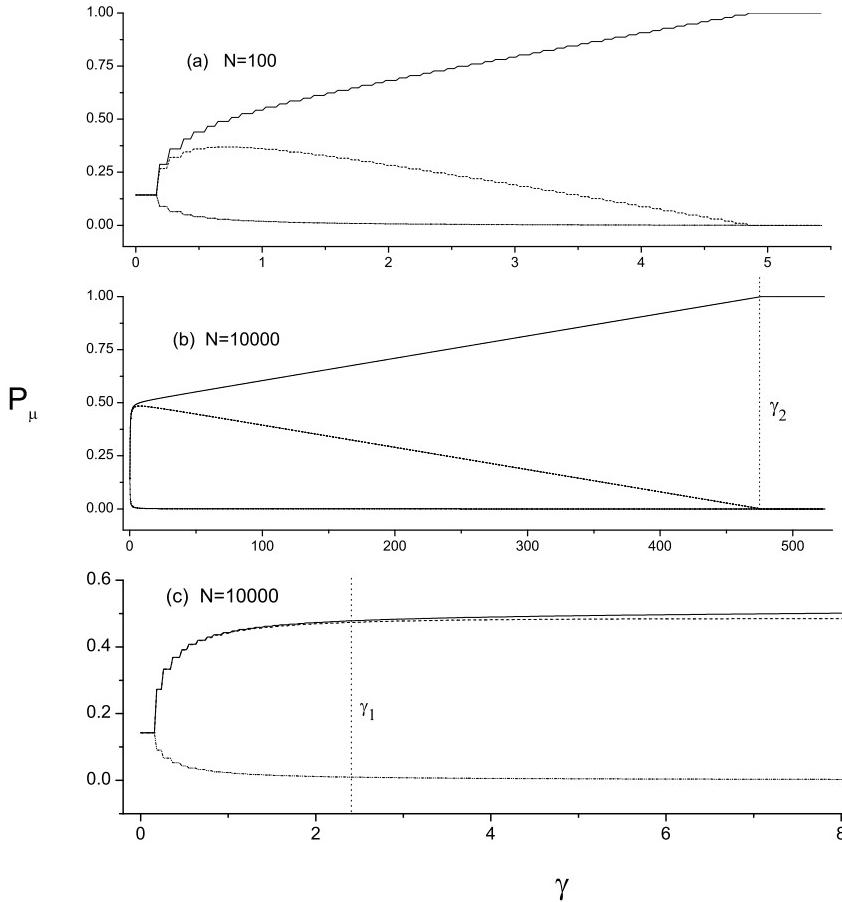


Fig. 1 P_μ (the probability of an atom in μ component) of the ground states of H'_{mod} against γ with $N = 100$ (a) and 10000 (b,c). P_3 is in solid line, P_{-3} in dash line, and all other P_μ are in dash-dot-dot line (they overlap exactly). (c) is just the left end of (b).

P_μ ($\mu \neq \pm 3$) are found to be equal to 0.01. We use the range 0 to γ_1 to define the first stage. The second stage is from γ_1 to $\gamma_2 = (2N + 3)/42$ (say, when $N = 10^4$, $\gamma_1 = 2.2$ and $\gamma_2 = 476.3$ as shown in Fig.1c and 1b). In this stage P_3 increases (P_{-3} decreases) nearly linearly until equal to 1 (0), while the other P_μ remain ≈ 0 . The third stage ($\gamma \geq \gamma_2$) is characterized by $P_3 = 1$, i.e., the system is fully polarized.

Incidentally, the structure of the ground states of the ^{52}Cr condensates under a magnetic field has already been studied by a number of authors using the mean field theory.[9, ?, ?] Obviously, the spinors from the mean field theory can be compared with P_μ . When g_0 is very negative, two phases "polar" and "ferromagnetic" have been found. The associated spinors are $(\cos \theta, 0, 0, 0, 0, 0, \sin \theta)$ and $(1, 0, 0, 0, 0, 0, 0)$, respectively. Obviously, the former (latter) corresponds to the above second (third) stage of P_μ . From the correspondence we know that the underlying physics of the "polar phase" is just

a kind of structure wherein a portion of particles are fully polarized while all other particles are in $s = 0$ pairs.

Using the concept of the fractional parentage coefficients one can extract any pair of particles, say i and j , from a spin-state $\Psi = \sum_k C_k |k\rangle$ as

$$\begin{aligned} \Psi = & \sum_{\mu,\nu} \chi_\mu(i) \chi_\nu(j) [\delta_{\mu\nu} \sum_k C_k \sqrt{\frac{N_\mu^k(N_\mu^k - 1)}{N(N-1)}} | \dots, N_\mu^k - 2, \dots \rangle \\ & + \bar{\delta}_{\mu\nu} \sum_k C_k \sqrt{\frac{N_\mu^k N_\nu^k}{N(N-1)}} | \dots, N_\mu^k - 1, \dots, N_\nu^k - 1, \dots \rangle] \end{aligned} \quad (20)$$

where the two Fock-states written explicitly at the right are for $(N-2)$ -body systems. The former originates from $|k\rangle$ by changing N_μ^k to $N_\mu^k - 2$, while the latter by changing both N_μ^k and N_ν^k to $N_\mu^k - 1$ and $N_\nu^k - 1$, respectively. From this expansion it is straight forward to calculate how the magnetization m of a selected pair would be. Let μ and ν denote the spin-components of the two selected atoms i and j , and $m = \mu + \nu$. Then, the probability of the pair in m reads

$$Q_m = \sum_{\mu,\nu} \delta_{\mu+\nu,m} [\delta_{\mu\nu} \sum_k C_k^2 \frac{N_\mu^k(N_\mu^k - 1)}{N(N-1)} + \bar{\delta}_{\mu\nu} \sum_k C_k^2 \frac{N_\mu^k N_\nu^k}{N(N-1)}]. \quad (21)$$

In addition to P_μ , Q_m is a measure of spin-correlation (corresponding to the two-body correlation function in spatial space), and has no counterpart in mean field theory. Q_m of the ground states of H'_{mod} against γ are plotted in Fig.2. Among all the thirteen Q_m (m is from -6 to 6), Q_0 is much larger than the others if $\gamma \leq 7/42$ (in this case all particles are in $s = 0$ pairs). Thus, for every pair of particles, the probability of having their spins anti-parallel is the largest. The next largest is the group $Q_1 = Q_2 = Q_{-1} = Q_{-2}$, while the smallest is the group $Q_6 = Q_5 = Q_{-6} = Q_{-5}$. However, when γ increases, Q_6 will increase nearly linearly. Q_0 and Q_{-6} increase firstly with γ but eventually decrease with γ as plotted. Other Q_m are very small. When $\gamma \geq \gamma_2$, $Q_6 = 1$ implying a full polarization as expected.

4 Summary

A model is proposed to study the possible pairing structures of N-boson systems with nonzero spin. Analytical solutions have been obtained. The following points are mentioned

When the pairing force is negative, the ground state has all particles forming the $s = 0$ pairs. The excited states are described by good quantum numbers S , M , and I . The latter is just the number of particles not in the $s = 0$ pairs. The state has a larger I is higher, and the states have the same I are degenerate. When a magnetic field B is applied, the ground state will have a portion of particles fully polarized, while the other particles remain in the $s = 0$ pairs.

When the pairing force is positive, the ground state is highly degenerate with a gap proportional to N . Therefore, when N is large, the spin-degrees of freedom are difficult to be excited.

Two quantities P_μ and Q_m have been defined and studied. They are the spin-space analogues of the one- and two-body densities of coordinate space. In particular, Q_m

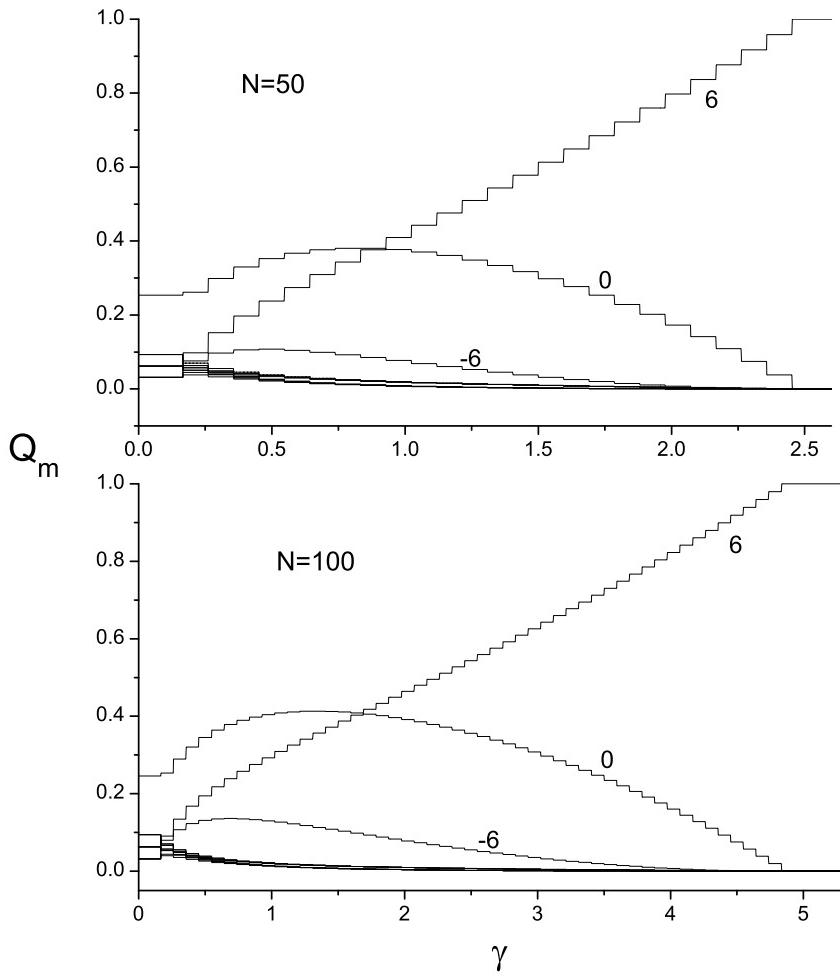


Fig. 2 Q_m (the probability of a pair of atoms with magnetization m) of the ground states of H'_{mod} against γ with $N = 50$ (a) and 100 (b). m is marked by the associated curves (the lowest a group of curves are not marked). Q_1 overlaps Q_2 , Q_{-1} overlaps Q_{-2} , Q_3 overlaps Q_4 , and Q_{-3} overlaps Q_{-4} .

can help us to understand better the spin-correlation and has no counterpart in mean field theory. Therefore, these quantities are in general useful for understanding the spin-structures of various systems. For an example, the underlying structure of the "polar phase" found in [9, ?, ?] is thereby clarified as a mixture of fully polarized particles and $s = 0$ pairs.

In general, the results from this model system might be helpful for the understanding of some realistic systems.

Acknowledgements The support from the NSFC under the grant 10574163 and 10874249 is appreciated.

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4. Stamper-Kurn, D.M., et al.: Phys. Rev. Lett. **80**, 2027 (1998).
5. In the theory of condensates based on the mean field theory, $f(\mathbf{r})$ depends on μ , the Z-component of the spin of an atom. When the dependence is omitted, it is called the single mode approximation (SMA). This approximation would be valid if the particle density is dilute, the temperature is very low, and the spin-dependent interaction is weak. This is just the case of most condensates in general. Therefore, the SMA is a good approximation and has been widely used in the literatures of condensates. In this paper the SMA is used also.
6. Although the energy gap of the ground state does not depend on N if $|G_0|$ is used as a unit, since $\int d\mathbf{r}|f(\mathbf{r})|^4$ is roughly proportional to $N^{-3/5}$, $|G_0|$ decreases slowly with the increase of N . Thus a larger N would actually lead to a smaller gap when the pairing force is attractive.
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